Quasi-thermal noise in space plasma: “kappa” distributions

G. Le Chat, K. Issautier, N. Meyer-Vernet, I. Zouganelis, M. Maksimovic, and M. Moncuquet

1LESA, Observatoire de Paris, CNRS, UPMC, Université Paris Diderot, 5 Place Jules Janssen, 92195 Meudon, France
2Laboratoire de Physique des Plasmas, UPMC, Ecole Polytechnique, CNRS, Univ. Paris 11, 4 avenue de Neptune, 94107 Saint-Maur-des-Fossés, France

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The transport of energy in collisionless plasmas, especially in space plasmas, is far from being understood. Measuring the temperature of the electrons and their nonthermal properties can give important clues to understand the transport properties. Quasi-thermal noise (QTN) spectroscopy is a reliable tool for measuring accurately the electron density and temperature since it is less sensitive to the spacecraft perturbations than particle detectors. This work models the plasma QTN using a generalized Lorentzian (“kappa”) distribution function for the electrons. This noise is produced by the quasi-thermal fluctuations of the electrons and by the Doppler-shifted thermal fluctuations of the ions. A sum of two Maxwellian functions has mainly been used for modeling the QTN of the electrons, but the observations have shown that the electrons are better fitted by a kappa distribution function.

Pioneer work on QTN calculation only considered integer values of \( \kappa \). This paper extends these calculations to real values of \( \kappa \) and gives the analytic expressions and numerical calculations of the QTN with a kappa distribution function. This paper shows some generic properties and gives some practical consequences for plasma wave measurements in space. © 2009 American Institute of Physics. [doi:10.1063/1.3243495]

I. INTRODUCTION

In collisionless plasma, like the solar wind, the mechanism of energy transport is still an open question. Due to the high difference of mass between the ions and the electrons, the electrons transport the energy whereas the ions transport the impulsion. Consequently, measuring accurately the temperature of the electrons and their nonthermal properties with quasi-thermal spectroscopy can give important clues to understand the energy transport properties.

By the same way as a passive electric antenna is sensitive to electromagnetic waves, it is also sensitive to local fluctuations of the electric potential. These fluctuations are produced by the motions of the ambient electrons and ions. As soon as the plasma is stable, this quasi-thermal noise (QTN) is completely determined by the particle velocity distributions in the frame of the antenna.

The problem is simpler in the absence of a static magnetic field or at frequencies much higher than the electron gyrofrequency, since in this case the plasma can be considered to be an assembly of “dressed test” particles moving in straight lines. The QTN spectrum around the plasma frequency \( f_p \) consists of a noise peak just above \( f_p \) produced by electron quasi-thermal fluctuations. Since the plasma density \( n_e \) is proportional to \( f_p^2 \), this allows an accurate measurement of the electron density. In addition, since the shape of the spectrum is determined by the electron velocity distribution, the analysis of the spectrum reveals its properties. One of the main advantages of the QTN spectroscopy is its relative immunity to the spacecraft potential and photoelectrons perturbations which, in general, affect particle analyzers.

A sum of two Maxwellian functions has mainly been used for modeling the electron velocity distributions. The observations have shown that the suprathermal electrons are better fitted by a generalized Lorentzian (“kappa”) distribution function, first introduced to model space data by Vasyliunas and Olbert. In this paper, we compute the QTN obtained for such electron distribution functions with real kappa parameter. Chateau and Meyer-Vernet only considered integer values of \( \kappa \) in order to simplify the calculation of the longitudinal dielectric permittivity. Section II shows how to generalize this calculation to real values of kappa and gives the corresponding QTN. In Sec. III, we present other contributions to the thermal noise in usual space plasmas. General properties of the QTN are shown in Sec. IV.

II. KAPPA ELECTRON THERMAL NOISE

A. Basics

The voltage power spectrum of the plasma QTN at the terminals of an antenna in a plasma drifting with velocity \( \vec{V} \) is

\[
V_e^2 = \frac{2}{(2\pi)^3} \int \frac{\vec{k} \cdot \vec{j}}{k} E^2(\vec{k}, \omega - \vec{k} \cdot \vec{V}) dk.
\]

The first term in the integral involves the antenna response to electrostatic waves, which depends on the Fourier transform \( \hat{J}(\vec{k}) \) of the current distribution along the antenna. The second term is the autocorrelation function of the electrostatic field fluctuations in the antenna frame. At frequencies much higher than the gyrofrequency, we have
$E^2(\vec{k},\omega) = 2\pi \sum_q q_j^2 \int f_j(\vec{v}) \delta(\omega - \vec{k} \cdot \vec{v}) d^3v \frac{k^2|\vec{v}_L(\vec{k},\omega)|^2}{k^2e_0^2e_L(\vec{k},\omega)^2}$, \hfill (2)

$f_j(\vec{v})$ being the velocity distribution of the $j$th species of charge $q_j$ and $e_L(\vec{k},\omega)$ the plasma longitudinal function. \hfill (3)

The longitudinal dielectric permittivity is given by Eq. (5). Considering a kappa function, we obtain

$$e_L = 1 + \frac{2\pi \omega_p^2}{k^2} A \int_{-\infty}^{+\infty} \frac{\phi(v)}{(k\nu - \omega - io)(1 + v^2/k^2\nu_0^2)^{\kappa+1}} dv.$$ \hfill (12)

Setting $x = v/\kappa^{1/2}v_0$ and $z = \omega/\kappa^{1/2}k\nu_0$, and using partial fraction decomposition of the integrand, we obtain

$$e_L = 1 + \frac{2\pi \omega_p^2}{k^2} \int_{-\infty}^{+\infty} \frac{dx}{(x^2 + 1)^{\kappa+1}}.$$ \hfill (13)

The integrand $I$ shows a pole for $x = z + io$. Setting

$$\frac{1}{(x^2 + 1)^{\kappa+1}} = \left[ \frac{1}{(x^2 + 1)^{\kappa+1}} - \frac{1}{(z^2 + 1)^{\kappa+1}} \right] + \frac{1}{(z^2 + 1)^{\kappa+1}},$$ \hfill (14)

we obtain

$$I = A v_0^{\kappa+1/2} \left\{ \int_{-\infty}^{+\infty} \frac{1}{(x^2 + 1)^{\kappa+1}} \right\},$$ \hfill (15)

The first integrand shows a multiplying factor that vanishes when $x = z$. It is analytic near the real axis and the term $io$ is no longer required. The second term is calculated by the residue theorem, indenting the contour about $z + io$ with a small semicircle and closing it by a large semicircle in the upper half-plane, and gives

$$\frac{i\pi}{(z^2 + 1)^{\kappa+1}}.$$ \hfill (16)

Thus the imaginary part of the dielectric permittivity is...
\[ \text{Im}[\varepsilon_L] = \frac{2\Gamma(k + 1)z^3}{\Gamma(k - 1/2)\pi^2} \left( z^2 + 1 \right)^{k-1}, \]  

(17)

where \( r = f/f_p = w/w_p \).

The first part of Eq. (16) will be numerically integrated taking care of the infinite upper and lower limits. Due to the term \( 1/(x-z) \), the integration needs to be separated in three parts,

\[
\int_{-\infty}^{\infty} \frac{1}{(x-z)} \left[ \frac{1}{(x^2 + 1)^{k+1}} - \frac{1}{(z^2 + 1)^{k+1}} \right] dx
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{(x-z)} \left[ \frac{1}{(x^2 + 1)^{k+1}} - \frac{1}{(z^2 + 1)^{k+1}} \right] dx
\]

\[
+ \int_{-\beta}^{\beta} \frac{1}{(x-z)} \left[ \frac{1}{(x^2 + 1)^{k+1}} - \frac{1}{(z^2 + 1)^{k+1}} \right] dx
\]

\[
+ \int_{\beta}^{\infty} \frac{1}{(x-z)} \left[ \frac{1}{(x^2 + 1)^{k+1}} - \frac{1}{(z^2 + 1)^{k+1}} \right] dx, \tag{18}
\]

where \( \beta > z \), so that

\[
- \frac{1}{(z^2 + 1)^{k+1}} \left[ \int_{-\infty}^{-\beta} \frac{1}{(x-z)} dx + \int_{\beta}^{\infty} \frac{1}{(x-z)} dx \right]
\]

\[
= - \frac{1}{(z^2 + 1)^{k+1}} \left[ \ln \left( \frac{\beta + z}{\beta - z} \right) \right].
\]

The remaining parts are proper integrals which can be calculated numerically after using Taylor’s expansion in l’Hôpital’s rule around \( x = z \). The longitudinal dielectric permittivity calculated by this method is valid for any values of kappa greater than 3/2. In the particular case when \( \kappa \) is integer, it is equivalent to the formula given by Ref. 7 and used in Refs. 11 and 20.

D. Kappa electron thermal noise

The fluctuations of the electrostatic field are given in Eq. (3). The dielectric function \( \varepsilon_L \) has just been calculated. \( F(kL) \) depends on the geometry of the antenna with expression (6) or expression (7) for, respectively, wire or sphere antennas, where \( L \) is the length of the antenna. Let us now calculate \( B(k) \).

Inserting Eq. (8) into Eq. (4), we get

\[
B(k) = \frac{2\pi \alpha}{\kappa} \int_{\omega_0}^{\infty} \frac{\varepsilon_0 \varepsilon_{\text{vac}}^2}{(1 + \varepsilon_0/k^2\varepsilon_{\text{vac}}^2)^{4k+1}} dv,
\]

which gives, after we set \( x = \varepsilon_0/k^2\varepsilon_{\text{vac}}^2 \) and \( z = \omega/\kappa^{1/2}v_0 \),

\[
B(k) = \frac{\pi \alpha \varepsilon_0^2}{\kappa} \frac{1}{(1 + z^3)^\kappa}. \tag{19}
\]

Substituting this expression of \( B(k) \) into Eq. (3) and setting \( r = \omega/\omega_p \), \( u = L/L_D \) with \( L_D \) given by Eq. (11), \( v_0 \) given by Eq. (10), and \( A \) given by Eq. (9), we find the expression of the normalized QTN spectrum,
\[ F_a(x) = j_0(x) \] (wires),
\[ F_a(x) = \left( \frac{\sin x}{x} \right)^2 \] (spheres),

where \( j_0(x) \) is the Bessel's function.

The relative contribution of this noise \( V_i^2/V^2 \) for \( f/f_p \) 
less than 1 is generally negligible for the wire dipole (since \( a < L_D \)),
but it is dominant for sphere antennas.\(^{10}\) This is the reason why QTN measurements on space missions used wire dipoles.
For wire antennas, this noise decreases with frequency as \( 1/f^2 \).

The antenna impedance (22) is the main contribution of the radio receiver impedance \( Z_R \).\(^{21}\) Hence the voltage power spectrum \( V_i^2 \), which is the interesting quantity, is related to that measured by the receiver \( V_R^2 \) by the relation
\[ V_R^2 = V_i^2 \left| \frac{Z_R}{Z_R + Z} \right|^2. \] (23)
Thus, a good determination of the antenna impedance is required for any accurate plasma measurement.

B. Ion thermal noise in drifting plasma

In the case of a drifting plasma where the thermal velocity \( v_{th} \) of the ion is smaller than the velocity of the plasma \( V \), like the solar wind, the lower-frequency part of the QTN spectrum is due to the above-mentioned shot noise and to the ion thermal noise, which is Doppler shifted by the plasma velocity. This noise has been extensively studied in Ref. 3. The ion contribution to the voltage power spectrum in cylindrical coordinates of axis parallel to \( \vec{V} \) is

\[ V_i^2 = \frac{8}{\pi^2} \frac{ne^2}{e_0 e_{th}} \int_0^{\pi/2} \frac{dk}{k} \times \int_{-1}^{1} \exp\left(\omega - kV_u^2 v_{th}^2 k^2\right) \left| \frac{e_i(k, \omega - kV_u^2)}{e_i(k, \omega)} \right|^2 du \times \int_0^{2\pi} \frac{\sin^3(\frac{kL \cos \gamma}{2 \cos \gamma})}{(kL \cos \gamma)^2} d\psi, \] (24)

where \( u = \cos \theta \) with \( \theta \) as the angle between \( \hat{k} \) and \( \vec{V} \), and \( \gamma \) is the angle between the antenna and \( \hat{k} \) given by
\[ \cos \gamma = u \cos \beta + \sqrt{1 - u^2} \sin \beta \cos \psi, \]
where \( \beta \) is the angle between \( \vec{V} \) and the antenna, and \( \psi \) is the azimuthal angle of \( \hat{k} \) in a plane perpendicular to \( \vec{V} \).

To deduce a simpler formula to be used for a plasma diagnostic, one could consider the two simple cases where the antenna is perpendicular or parallel to the velocity \( \vec{V} \).\(^3\)
The first case is the most interesting in practice. Since the effect of the velocity is maximum for \( \hat{k} \) parallel to \( \vec{V} \) and since for a long antenna the maximum response is ultimately at 90° from the antenna direction,\(^{22}\) the ion contribution to the thermal noise (which increases with the Doppler shift) is expected to be maximum when the antenna is perpendicular to \( \vec{V} \).
nomic behavior of QTN spectra: a plateau below $f_p$, a cutoff at $f_p$ with a peak, which is sharper for longer antennas, and a high-frequency spectrum proportional to $f^{-3}$. The spectrum is nearly flat for short antennas. For longer antennas, the peak is sharp and occurs at $f_p$, which allows a very precise measurement of $n_e$ since the electron density is proportional to $f_p^2$.

Figure 4 shows the effect of changing the parameter $\kappa$ for a wire dipole antenna with two different values of the antenna length, 4 and 64 m corresponding to, respectively, 0.5 and 8 times the Debye length for $\kappa=3.5$, $n=5.6 \times 10^6$ m$^{-3}$, and $T_p=1.12 \times 10^5$ K. Thus, the spectrum with $\kappa=0.5$ and $\kappa=8$ is equivalent in Figs. 3 and 4. The low-frequency level does not depend very much on $\kappa$ for $\kappa \gg 1$, especially for shorter antennas. This is because it depends mainly on the bulk of the velocity distribution.$^{9,10}$ The effect of varying $\kappa$ is stronger close to the plasma frequency. We also see that for a long wire antenna the high-frequency level depends not on $\kappa$, but only on $n$ and $T_p$. At frequency close to the plasma frequency, the phase velocity of Langmuir waves becomes very large, since the wave number is zero at $f_p$. Consequently, the fastest particles will resonate with these waves, producing the noise peak at $f=f_p$. Since the $\kappa$ index is related to the proportion of suprathermal electrons, its high influence on the QTN spectrum at the plasma frequency is easily explained.

V. CONCLUSIONS

We developed the numerical solution for the determination of the longitudinal dielectric permeability in the case of a kappa distribution function for all values of $\kappa$ and applied it to the QTN spectroscopy. We derived the following properties of the electron QTN in this case. The $\kappa$ parameter’s main influence takes place close to the plasma frequency and above. The high-frequency ($f \gg f_p$) noise level on a wire antenna of length $L \gg L_D$ only depends on the electron density ($n$) and temperature ($T_p$), and the analytic expression

$$ V^2(\kappa) = 4 \times 10^{-11} T_p(K) n(m^{-3}) f_p^3(\text{Hz}) L(m) $$

given by Chateau and Meyer-Vernet$^7$ can be used to obtain an approximation of the noise level. Therefore the measurement of this level gives a direct determination of the pressure for any stable distribution function.

Figure 5 summarizes the parameters one could expect to determine with the QTN spectroscopy in a drifting and collisionless plasma, like the solar wind. In a more practical view, the calculation of the QTN with any value of the parameter $\kappa$ allows us to analyze the data provided by space missions in order to obtain precise measurements of the electron density, temperature, and nonthermal parameters in the solar wind, which is essential to understand the energy transport in collisionless plasma.